

# Eigenvalues of symmetric tridiagonal interval matrices revisited

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## Abstract

In this short note, we present a novel method for computing exact lower and upper bounds of a symmetric tridiagonal interval matrix. Compared to the known methods, our approach is fast, simple to present and to implement, and avoids any assumptions. Our construction explicitly yields those matrices for which particular lower and upper bounds are attained.

**Keywords:** *Eigenvalue; Tridiagonal matrix; Interval matrix.*

## 1 Introduction

Consider a tridiagonal symmetric matrix of size  $n$

$$A = \begin{pmatrix} a_1 & b_2 & 0 & \dots & 0 \\ b_2 & a_2 & b_3 & \ddots & \vdots \\ 0 & b_3 & a_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_n \\ 0 & \dots & 0 & b_n & a_n \end{pmatrix}.$$

and denote by  $\lambda_1(A) \geq \dots \geq \lambda_n(A)$  its eigenvalues. Assume that the entries of  $A$  are not known precisely and the only information that we have is that  $a_i$  comes from a given interval  $\mathbf{a}_i = [\underline{a}_i, \bar{a}_i]$ ,  $i = 1, \dots, n$ , and  $b_i$  comes from a given interval  $\mathbf{b}_i = [\underline{b}_i, \bar{b}_i]$ ,  $i = 2, \dots, n$ .

By  $\mathbf{A} = [\underline{\mathbf{A}}, \bar{\mathbf{A}}]$  we denote the corresponding interval matrix, that is, the set of all matrices when  $a_i \in \mathbf{a}_i$  and  $b_i \in \mathbf{b}_i$ . Next,

$$\lambda_i = \{\lambda_i(A); A \in \mathbf{A}\}$$

stands for the corresponding eigenvalue sets. It was shown in [8] that they form real compact intervals. The problem investigated in this paper is to determine their end-points. We focus on the upper end-points  $\bar{\lambda}_i$ s since the lower ones can be determined analogously by the reduction  $A \mapsto -A$ .

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Characterization of the extremal eigenvalues  $\bar{\lambda}_1$  and  $\underline{\lambda}_n$  of a general symmetric interval matrix is due to Hertz [6] by a formula involving computation of  $2^n$  matrices. A partial characterization of the intermediate eigenvalue intervals was done in [9, 5]. Due to NP-hardness of computing or even tightly approximating the eigenvalue sets [7, 15], there were developed various outer and inner approximation methods [8, 9, 1, 10, 11, 12, 14, 16]. The tridiagonal case was particularly investigated by Commerçon [3], who proposed a method for calculating the exact eigenvalue bounds. This method, however, suffer from simplicity, time complexity analysis and generality. In what follows, we overcome all these drawbacks.

## 2 Our method

**Proposition 1.** *Without loss of generality, we can assume that  $A \geq 0$ .*

*Proof.* The transformation  $A \mapsto A + \alpha I_n$  increases all eigenvalues of  $A$  by the amount of  $\alpha$ . So for any  $\alpha \geq \max_i \{-a_i\}$  this transformation yields a matrix with a nonnegative diagonal. Thus, we can assume that  $a_i \geq 0$  for every  $i$ .

Suppose now there is  $i$  such that  $b_i < 0$ . Let  $\lambda$  be any eigenvalue of  $A$  and  $x$  a corresponding eigenvector, that is,  $Ax = \lambda x$ . Let  $A'$  be the matrix resulting from  $A$  by putting  $b'_i = -b_i$ , and let  $x' = (-x_1, \dots, -x_{i-1}, x_i, \dots, x_n)^T$ . Then for  $k > i$  we have

$$(A'x')_k = (Ax)_k = (\lambda x)_k = (\lambda x')_k.$$

For  $k < i - 1$  we have

$$(A'x')_k = (-Ax)_k = (-\lambda x)_k = (\lambda x')_k.$$

The remaining two cases are:

$$(A'x')_{i-1} = b'_{i-1}x'_{i-2} + a'_{i-1}x'_{i-1} + b'_i x'_i = -b_{i-1}x_{i-2} - a_{i-1}x_{i-1} - b_i x_i = -\lambda x_{i-1} = \lambda x'_i,$$

and

$$(A'x')_i = b'_i x'_{i-1} + a'_i x'_i + b'_{i+1} x'_{i+1} = b_i x_{i-1} + a_i x_i + b_{i+1} x_{i+1} = \lambda x_i = \lambda x'_i.$$

Thus,  $A'$  has the same eigenvalues as  $A$ , and the eigenvectors of  $A$  can easily be derived from those of  $A'$ . By repeating this process, we obtain all  $b_i$ s nonnegative.  $\square$

We can therefore assume that  $\underline{A} \geq 0$  for the interval matrix  $\mathbf{A}$ . Nonnegativity of the diagonal can be achieved by the transformation  $\mathbf{a}_i \mapsto \mathbf{a}_i + \alpha$  with  $\alpha := \max_i \{-\underline{a}_i\}$ , and nonnegativity of the remaining entries by the transformation

$$\mathbf{b}_i \mapsto \begin{cases} \mathbf{b}_i, & \text{if } \underline{b}_i \geq 0, \\ -\mathbf{b}_i, & \text{if } \bar{b}_i \leq 0, \\ [0, \max\{-\underline{b}_i, \bar{b}_i\}], & \text{otherwise.} \end{cases}$$

Suppose  $\bar{b}_i > 0$  for all  $i = 2, \dots, n$ , otherwise we split the problem into the sub-problems corresponding to the diagonal blocks of  $\mathbf{A}$ .

Suppose now that  $\underline{b}_i > 0$  for all  $i = 2, \dots, n$ .

Let  $\lambda(A)$  be an eigenvalue of  $A$  and  $x(A)$  a corresponding eigenvector. They can be chosen in such a way that they constitute continuous mappings with respect to  $A \in \mathbf{A}$ . We say that the eigenvectors of  $\mathbf{A}$  are *sign invariant* [4] if  $x_i(A) \neq 0$  for all  $A \in \mathbf{A}$  and all  $i$ s. By other words, either  $x_i(A) > 0$  for every  $A \in \mathbf{A}$ , or  $x_i(A) < 0$  for every  $A \in \mathbf{A}$ .

**Proposition 2.** *Suppose that  $\underline{b}_i > 0$  for all  $i$ s. Then all eigenvalues of every  $A \in \mathbf{A}$  are simple and the eigenvectors are sign invariant.*

*Proof.* The first part is obvious since it is known that a symmetric diagonal matrix has simple eigenvalues provided off-diagonal elements are nonzero [13].

For the second part, suppose to the contrary that there is  $A \in \mathbf{A}$  such that its eigenvector  $x$  has some zero entries. Let  $\lambda$  be the corresponding eigenvalue. Without loss of generality assume that  $x$  has the form of  $x^T = (0^T, y^T) = (0, \dots, 0, y_1, \dots, y_k)$ , where  $y_i \neq 0$  for all  $i = 1, \dots, k$ . Decompose  $A$  into blocks accordingly

$$A = \begin{pmatrix} B & C \\ C^T & D \end{pmatrix},$$

where  $B \in \mathbb{R}^{(n-k) \times (n-k)}$ ,  $C \in \mathbb{R}^{(n-k) \times k}$  and  $D \in \mathbb{R}^{k \times k}$ . The equality  $Ax = \lambda x$  implies  $B0 + Cy = \lambda 0$ , whence  $Cy = 0$ . Since

$$C = \begin{pmatrix} 0 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \\ b_{n-k+1} & 0 & \dots & 0 \end{pmatrix},$$

and by the definition of  $y$ , we have  $b_{n-k+1} = 0$ . A contradiction.  $\square$

## The method

The derivative of a simple eigenvalue  $\lambda$  of a symmetric  $A$  according to  $a_{ij}$  is equal to  $x_i x_j$ , where  $x$ ,  $\|x\|_2 = 1$ , is the corresponding eigenvector. The derivative is nonnegative with respect to the diagonal entries of  $A$ , so the largest eigenvalues of  $\mathbf{A}$  are attained for  $a_i := \bar{a}_i$ . Due to sign invariancy of eigenvectors, we can easily determine also  $b_i$ -s. Let  $\lambda_k$  be the  $k$ th largest eigenvalue of  $A_c$  and  $x$  the corresponding eigenvector. Define

$$b_i := \begin{cases} \bar{b}_i & \text{if } x_i x_{i+1} > 0, \\ \underline{b}_i & \text{otherwise.} \end{cases}$$

Then  $\bar{\lambda}_k$  is attained as the  $k$ th eigenvalue of the matrix in this setting.

In particular, from the Perron theory and properties of nonnegative matrices, we have that  $\bar{\lambda}_1$  is attained for  $A := \bar{A}$ .

Suppose now the general case, that is, there might be some  $i$  such that  $\underline{b}_i = 0$ . Consider the matrix  $\mathbf{A}_\varepsilon$  such that  $\mathbf{b}_i = [\varepsilon, \bar{b}_i]$  for all such  $i$ s and  $\varepsilon > 0$  sufficiently small. This matrix  $\mathbf{A}_\varepsilon$  satisfies the above assumptions, so we can determine the eigenvalues sets accordingly. From the continuity reasons, as  $\varepsilon \rightarrow 0$ , the matrix  $\mathbf{A}_\varepsilon$  converges to  $\mathbf{A}$ . Therefore, the above procedure can be applied even in the general case. The sign invariancy then holds in a weaker sense that some entries of eigenvectors may vanish, but do not change their sign.

As a side effect, we have the following interesting property.

**Proposition 3.**  $\bar{\lambda}_k$  is attained for  $a_i := \bar{a}_i$  and  $b_i \in \{\underline{b}_i, \bar{b}_i\}$  such that the cardinality of

$$\{i; b_i = \bar{b}_i\}$$

is  $n - k$ .

*Proof.* Let  $A \in \mathbf{A}$ , let  $\lambda_k$  be its  $k$ th eigenvalue and  $v$  a corresponding eigenvector. By [13], the sign of  $v_j$  is equal to the sign of

$$\chi_{j-1}(\lambda_k) b_{j+1} \dots b_n,$$

where  $\chi_{j-1}$  is the characteristic polynomial of the (top left) principal leading submatrix of  $A$  of size  $j - 1$ , and  $\chi_0 \equiv 1$ . Since  $b \geq 0$ , the signs of  $v_j$  and  $\chi_{j-1}(\lambda_k)$  coincide. The number of sign agreements between consecutive terms in the Sturm sequence  $\{\chi_i(\lambda_k); i = 0, 1, \dots, n\}$  gives the number of roots of  $\chi_n$  which are less than  $\lambda_k$ , that is  $n - k$ . Therefore, by the analysis of our method,  $n - k$  is equal to the number of  $b_i$ s that we set to the right end-point.  $\square$

As a simple corollary we get that  $\bar{\lambda}_1$  is attained for  $b := \bar{b}$  and  $\bar{\lambda}_n$  is attained for  $b := \underline{b}$ .

An interval matrix  $\mathbf{A}$  is called regular if every  $A \in \mathbf{A}$  is nonsingular; see [17]. Since we can determine the eigenvalue sets exactly, we can also decide whether 0 is an eigenvalue of some matrix. Therefore, we have that checking regularity of a tridiagonal symmetric interval matrix is a polynomial problem, which is a symmetric analogy of the results from [2].

Time complexity of our algorithm is the following. We need computation of eigenvalues of the midpoint matrix  $A_c$ , then  $2n$ -times computation of a certain eigenvalue of a matrix in  $\mathbf{A}$ . The preprocessing requires only linear time. Provided we employ a standard method for computation of eigenvalues of a real symmetric diagonal matrix running in  $\mathcal{O}(n^2)$ , the overall complexity is  $\mathcal{O}(n^3)$ .

As a consequence, we have a quadratic time method for testing the following properties of a symmetric tridiagonal interval matrix  $\mathbf{A}$ :

- positive (semi)-definiteness, i.e., whether each  $A \in \mathbf{A}$  is positive (semi)-definite;
- Schur or Hurwitz stability, i.e., whether each  $A \in \mathbf{A}$  is stable;
- spectral radius, i.e., the largest spectral radius over  $A \in \mathbf{A}$ .

**Example 1.** Consider the example from [8, 10, 12, 14]:

$$\mathbf{A} = \begin{pmatrix} [2975, 3025] & [-2015, -1985] & 0 & 0 \\ [-2015, -1985] & [4965, 5035] & [-3020, -2980] & 0 \\ 0 & [-3020, -2980] & [6955, 7045] & [-4025, -3975] \\ 0 & 0 & [-4025, -3975] & [8945, 9055] \end{pmatrix}.$$

First, we transform the matrix into a nonnegative one

$$\mathbf{A} = \begin{pmatrix} [2975, 3025] & [1985, 2015] & 0 & 0 \\ [1985, 2015] & [4965, 5035] & [2980, 3020] & 0 \\ 0 & [2980, 3020] & [6955, 7045] & [3975, 4025] \\ 0 & 0 & [3975, 4025] & [8945, 9055] \end{pmatrix}.$$

The eigenvalues of the midpoint matrix are  $\lambda_1 = 12641$ ,  $\lambda_2 = 7064.5$ ,  $\lambda_3 = 3389.9$ ,  $\lambda_4 = 905.17$ , and the corresponding eigenvectors are

$$\begin{aligned} v_1 &= (0.05575, 0.26874, 0.64725, 0.71116)^T, & v_2 &= (-0.3546, -0.7206, -0.2595, 0.5363)^T, \\ v_3 &= (0.71884, 0.14012, -0.55442, 0.39531)^T, & v_4 &= (0.59535, -0.62357, 0.45425, -0.22446)^T. \end{aligned}$$

Based on the signs of the entries of these vectors we can directly conclude that  $\bar{\lambda}_1$  is attained for  $\bar{A}$ , and similarly  $\bar{\lambda}_2, \bar{\lambda}_3, \bar{\lambda}_4$  are attained as the corresponding eigenvalues of the matrices

$$\begin{pmatrix} 3025 & 2015 & 0 & 0 \\ 2015 & 5035 & 3020 & 0 \\ 0 & 3020 & 7045 & 3975 \\ 0 & 0 & 3975 & 9055 \end{pmatrix}, \begin{pmatrix} 3025 & 2015 & 0 & 0 \\ 2015 & 5035 & 2980 & 0 \\ 0 & 2980 & 7045 & 3975 \\ 0 & 0 & 3975 & 9055 \end{pmatrix}, \begin{pmatrix} 3025 & 1985 & 0 & 0 \\ 1985 & 5035 & 2980 & 0 \\ 0 & 2980 & 7045 & 3975 \\ 0 & 0 & 3975 & 9055 \end{pmatrix},$$

respectively. Similarly we proceed for calculating the lower end-points of the eigenvalue sets. Eventually, we obtain the following exact eigenvalue sets (by using outward rounding)

$$\begin{aligned} \lambda_1 &= [12560.8377, 12720.2273], & \lambda_2 &= [7002.2827, 7126.8283], \\ \lambda_3 &= [3337.0784, 3443.3128], & \lambda_4 &= [842.9250, 967.1083], \end{aligned}$$

### 3 Conclusion

We presented a simple and fast algorithm for computing the eigenvalue ranges of interval symmetric matrices. Impreciseness of measurement and other kinds of uncertainty are often represented in the form of intervals. Therefore, checking various kinds of stability of uncertain systems naturally leads to the problem of determining eigenvalues of interval matrices. In this short note, we improved the time complexity and the overall exposition of the known method for the tridiagonal matrix case.

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